TRINITY COLLEGE

# Year 11 Mathematics Specialist
# Test 5 2019

Calculator Free
**Proof by Induction and Complex Numbers**

**STUDENT'S NAME** SOLUTIONS

**DATE**: Wednesday 25th September **TIME:** 50 minutes **MARKS**: 46

**INSTRUCTIONS:**
Standard Items: Pens, pencils, drawing templates, eraser

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1. (3 marks)

State the following recurring decimal as a fraction. It is not necessary to simplify the fraction.

1.28333333333…

Let $x = 1.283\dot{3}$

$1000x = 1283.33\dot{3}$

$-\quad 100x = 128.3\dot{3}$

$900x = 1155$

$x = \dfrac{1155}{900}$

2. (2 marks)

For the complex number $z = 3i - 2$, state:

(a) Re($z$) [1]

$= -2$

(b) $\overline{z}$ [1]

$= -2 - 3i$

3. (3 marks)

Determine the complex solutions to the equation $2x^2 - 4x + 7 = 0$ in their most simplified form.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(7)}}{2(2)}$$

$$x = \frac{4 \pm \sqrt{16 - 56}}{4}$$

$$x = 1 \pm \sqrt{-40}$$

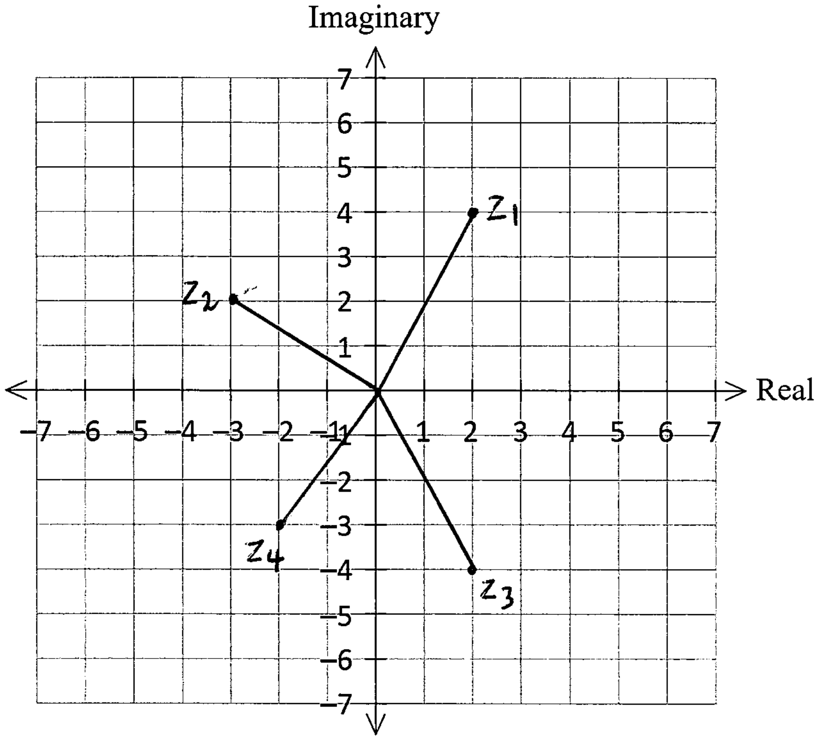
$$x = 1 \pm \frac{\sqrt{10}}{2} i$$

4. (4 marks)

Plot the following complex numbers on the argand diagram below. Label all the points clearly.

(a) $z_1 = 2 + 4i$ (b) $z_2 = -3 + 2i$

(c) $z_3 = \overline{z_1}$ (d) $z_4 = iz_2$




5. (4 marks)

If 3 - 2i is a root of the quadratic equation $x^2 + bx + c = 0$ , determine the values of b and c.

Real:

$3 = \frac{-b}{2}$

$6 = -b$

$b = -6$

Imaginary:

$-2i = \frac{\sqrt{36-4(c)}}{2}$

$-4i = \sqrt{36-4c}$

$-16 = 36-4c$

$-52 = -4c$

$13 = c$

6. (7 marks)

If $z = 2 - 5i$ and $w = -3 + 2i$, determine:

(a) $z - 2w$ [2]

$= (2 - 5i) - (-6 + 4i)$

$= 8 - 9i$

(b) $\frac{w}{z}$ [3]

$= \frac{-3+2i}{2-5i} \times \frac{2+5i}{2+5i}$

$= \frac{-6-15i+4i+10i^2}{4+25}$

$= \frac{-16-11i}{29}$

(c) $w\overline{w}$ [2]

$= 9 + 4$

$= 13$

7. (4 marks)

Determine $a$ and $b$ if $\frac{(1-3i)^2}{2-i} = a+bi$.

$$\frac{1-3i-3i+9i^2}{2-i} = a+bi$$

$$\frac{-8-6i}{2-i} \times \frac{2+i}{2+i} = a+bi$$

$$\frac{-16-8i-12i+6}{4+1} = a+bi$$

$$\frac{-10-20i}{5} = a+bi$$

$$-2-4i = a+bi$$

$\therefore$ $a = -2$

$b = -4$

8. (6 marks)

Prove, by mathematical induction, that $4^n - 1$ is divisible by 3 for any positive integer $n$.

Let $P(n)$ be the statement $4^n - 1 = 3A$ $\forall n, n \in \mathbb{Z}^+, A \in \mathbb{Z}$

$P(1)$: LHS $= 4^1 - 1$ $= 3$ RHS $= 3(1)$ $= 3$ ✓

$\therefore$ $P(1)$ is true

Assume $P(k)$ is true i.e. $4^k - 1 = 3A$, where $n \in \mathbb{Z}^+$ & $A \in \mathbb{Z}^+$

consider $P(k+1)$ to prove $4^{k+1} - 1$ is a multiple of 3 ✓

LHS $= 4^{k+1} - 1$ ✓

$= 4(4^k) - 4 + 3$

$= 4(4^k - 1) + 3$ ✓

$= 4(3A) + 3$

$= 3(4A+1)$ which is divisible by 3 ✓

$\therefore$ $P(k) \Rightarrow P(k+1)$, $P(1)$ is true ✓

$\therefore$ $P(n)$ is true $\forall n, n \in \mathbb{Z}^+$

9. (6 marks)

Use mathematical induction to prove the following conjecture:

$$1+x+x^2+x^3+\ldots+x^{n-1}=\frac{1-x^n}{1-x}, \quad n \geq 1, n \text{ a counting number.}$$

let $P(n)$ be the statement: $1+x+x^2+x^3+\ldots+x^{n-1} = \frac{1-x^n}{1-x}$, $\forall n, n\in$

$P(1)$: LHS $= x^{1-1}$ $\qquad$ RHS $= \frac{1-x^1}{1-x}$

$= x^0$ $\qquad$ $= 1$

$= 1$

$\therefore P(1)$ is true

Assume $P(k)$ is true $\forall k, k \in \mathbb{Z}$ i.e $1+x+x^2+\ldots x^{k-1} = \frac{1-x^k}{1-x}$

consider $P(k+1)$ to prove $1+x+x^2+\ldots+x^{k-1}+x^{(k+1)-1} = \frac{1-x^{k+1}}{1-x}$

$$\text{LHS} = 1+x+x^2+\ldots+x^{k-1}+x^{(k+1)-1}$$

$$= P(k) + x^{(k+1)-1}$$

$$= \frac{1-x^k}{1-x} + x^k$$

$$= \frac{1-x^k + (1-x)x^k}{1-x}$$

$$= \frac{1-x^k+x^k-x^{k+1}}{1-x}$$

$$= \frac{1-x^{k+1}}{1-x}$$

$$= \text{RHS}$$

$\therefore P(k) \Rightarrow P(k+1)$, $P(1)$ is true

$\therefore P(n)$ is true $\forall n, n \in \mathbb{Z}^+$

10. (6 marks)

Use mathematical induction to prove the following conjecture:

$$\frac{1}{1(3)}+\frac{1}{2(4)}+\frac{1}{3(5)}+\ldots+\frac{1}{n(n+2)}=\frac{3}{4}-\frac{2n+3}{2(n+1)(n+2)},\quad n \geq 1, n \text{ a counting number.}$$

let $P(n)$ be the statement $\frac{1}{1(3)}+\frac{1}{2(4)}+\ldots+\frac{1}{n(n+2)}=\frac{3}{4}-\frac{2n+3}{2(n+1)(n+2)}$

$P(1)$: $LHS = \frac{1}{1(1+2)} = \frac{1}{3}$

$RHS = \frac{3}{4} - \frac{2(1)+3}{2(1+1)(1+2)} = \frac{3}{4} - \frac{5}{12} = \frac{1}{3}$ ✓

$\therefore P(1)$ is true

Assume $P(k)$ is true $\forall K, K \in \mathbb{Z}^+$. i.e. $\frac{1}{1(3)}+\frac{1}{2(4)}+\ldots+\frac{1}{K(k+2)} = \frac{3}{4}-\frac{2k+3}{2(k+1)(K+2)}$

consider $P(K+1)$ to prove $\frac{1}{1(3)}+\frac{1}{2(4)}+\ldots+\frac{1}{K(k+2)}+\frac{1}{(k+1)((k+1)+2)} = \frac{3}{4}-\frac{2(k+1)+3}{2((k+1)+1)((k+1)+2)}$

$LHS = \frac{1}{1(3)}+\frac{1}{2(4)}+\ldots+\frac{1}{K(K+2)}+\frac{1}{(k+1)((k+1)+2)}$ ✓

$= P(k) + \frac{1}{(k+1)(k+3)}$

$= \frac{3}{4} - \frac{2k+3}{2(k+1)(k+2)} + \frac{1}{(k+1)(k+3)}$

$= \frac{3}{4} + \frac{2(K+2)-(2k+3)(k+3)}{2(k+1)(k+2)(k+3)}$

$= \frac{3}{4} + \frac{2k+4-2k^2-9k-9}{2(k+1)(k+2)(k+3)}$ ✓

$= \frac{3}{4} + \frac{-2k^2-7K-5}{2(k+1)(k+2)(k+3)}$

$= \frac{3}{4} - \frac{(2k+5)(k+1)}{2(k+1)(k+2)(k+3)}$ ✓

$= \frac{3}{4} - \frac{2k+5}{2(k+2)(k+3)}$ ✓

$= RHS$

$RHS = \frac{3}{4} - \frac{2(k+1)+3}{2((k+1)+1)((k+1)+2)}$

$= \frac{3}{4} - \frac{2k+2+3}{2(K+2)(k+3)}$

$= \frac{3}{4} - \frac{2k+5}{2(k+2)(k+3)}$ ✓

$\therefore P(k) \Rightarrow P(k+1)$, $P(1)$ is true

$\therefore P(n)$ is true $\forall n, n \in \mathbb{Z}^+$